WHAT IN HIGH SCHOOL MATHEMATICS IS OF MOST IMPORTANCE AS A PREPARATION FOR ANALYTIC GEOMETRY AND THE CALCULUS IN COLLEGE.

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One of the educational problems of the day, demanding solution with increasing insistence, is the question of the inter-relations of diverse college entrance examinations and the complex curriculum of the modern high school. This general problem is immediately suggested when a college instructor proposes to indicate the portions of the mathematical subjects commonly taught in the high school which are of most importance for the study of analytic geometry and the calculus. For such an indication implies that the colleges would like to have these portions of the elementary branches emphasized in the high school, and I can almost hear some high school instructor, on looking over the program, say to himself, "An attempt of the college to impose further unreasonable demands on the high school." Far be it from me to consider the topic proposed in any such spirit. Rather, let the discussion be conducted on the lines, fostered by this and other similar associations, along which progressive high school and college instructors are working harmoniously to secure in the high school a curriculum adapted to the needs of the large body of students who do not go to college and to the adequate preparation of the comparatively few who do.

With regard to the general problem mentioned above I shall maintain a discreet silence, my only reason for referring to it at all being to express the hope and belief that a frank discussion of the question under consideration, and other similar questions, will ultimately be of use in the solution of the general problem. That this discussion may be of such use is clear For if the topics taught in the high school of major importance to the pursuit of analytics and calculus have been listed, the remaining topics are of relatively little or perhaps of no impor

tance in that connection. If in the latter list there appear topics which are of little significance for the student not preparing for college, and if these topics are commonly included in college entrance requirements, I believe that the high schools are justified in asking that such topics be dropped from the entrance requirements. To this some college instructors may object that while certain topics are not used in analytics and calculus, nevertheless they are important in the development of an effective mathematical technique. But I think that it is entirely possible that this essential technique may be developed as well by other topics which may be of greater importance to the student whose formal education is completed by the high school course.

If anything further is needed to indicate that I do not approach this discussion with the thought that it affords an opportunity to criticize instruction in the secondary schools, let me say that I believe that many secondary schools, are teaching mathematics very effectively. As evidence, if any be needed, I would offer the fact that each year I find that one of the two best students in the sophomore course in analytics and calculus is a Freshman who has anticipated the mathematics of the Freshman year by courses in the high school. Let me also point out in this connection that in so far as the discussion relates to solid geometry, trigonometry and advanced algebra, it affects colleges, but not technical schools, quite as much as the secondary schools.

Although it would be possible and profitable to spend the time allotted me in answering the proposed question in general terms, I believe that it is more desirable to consider questions of detail. In what follows the term "important" refers, of course, to the preparation for analytics and calculus. Thus the graphical representation of complex numbers is of no importance in this connection but it is fundamental from other viewpoints.

Let us begin with algebra. It is almost superfluous to say that the utmost importance attaches to the four fundamental operations of addition, subtraction, multiplication and division. If these operations be applied to any set of numbers, whether they be denoted by a, b, c, and x, y, z or by more complicated symbols such as $\sqrt{ax+b}$, $(ax^2+bx+c)^{-3/2}$, 2^x, $\log x$, $\sin 2a$,

and arc sin $(a+b)$ being immaterial, we generally obtain fractions, and the problem of the simplification of complex fractions arises. An essential for the study of calculus is the ability to simplify fractions which may be denoted, when compared to the complex fractions in our elementary texts, as hyper-complex. I am restrained from giving way to the feelings aroused by the errors of the poorer students out of the fulness of my own experience. Even after I had been in college some time I approached a fraction with a respectful reluctance. After finishing the first course in calculus I turned to the algebra I had studied to look at the complex fractions which I remembered with something akin to horror, but I was unable to find them. After convincing myself that I had in hand the book I had studied and that there were no pages missing, I was amazed to discover that nearly all the complex fractions in the book could be simplified by inspection. Instead, therefore, of urging a drill in the simplification of more complicated fractions, I believe that what is of importance is a thorough grounding in fundamental principles. As the student matures mathematically his technique should improve so that by the time it is necessary to handle complicated cases he can do so.

No one would question the importance of simple cases of factoring such as ax^2+bx+c, including perfect squares and the difference of two squares, the sum and difference of two cubes, and an expression of four terms factorable by grouping or by regarding it as the difference of two squares or as a perfect cube. Other forms such as $x^4+x^2y^2+y^4$, $a^n \pm b^n$; and expressions containing five or six terms are unnecessary. And yet I recall one text which employs the quotient of $a^n - b^n$ by $a-b$ in obtaining the derivative of x^n for integral n. But the binomial theorem can be used instead, or, better still, the desired result may be obtained by repeated application of the fundamental rule for differentiating a product.

As I have already indicated, one should be familiar with the application of the fundamental operations to radicals, and, in addition, with the extraction of the square root of a number and with the rationalization of numerator or denominator of such functions as $\frac{a-\sqrt{x}}{a+\sqrt{x}}$ and $\sqrt{\frac{a-x}{a+x}}$. The extraction of the square root of a binomial surd is of no importance.

As regards the theory of exponents the laws for $a^m a^n$, $a^m \div a^n$, $(a^m)^n$, a^0, a^{-m}, where a may be any function, are important. But many problems in our texts and on college entrance examinations are not of a kind the student meets later, although they may be of value in developing or showing his understanding of the laws.

Another essential is the theory of logarithms as given in our texts exclusive of the rule for change of base. The solution of exponential equations is useful in a very few problems and also in so far as it shows how to take the logarithm of both sides of an equation of the form $a^b = c$, an important process.

Turning now to the fundamental question of algebraic equations, it is unnecessary to say that the solution of one or more equations which can be reduced by the removal of fractions and radicals to linear and quadratic equations, numerical or literal, is indispensable. In addition, it is desirable that the student should be able to determine whether two linear equations in two variables have one solution, are inconsistent, or have an infinite number of solutions. And in the last case, he should be able to find as many solutions as may be necessary. The solution of a system of linear equations, homogeneous as well as non-homogeneous, the conditions for a solution, and the elimination of the variable from such a system, by means of determinants might be applied very frequently and to great advantage in analytics, and occasionally in the calculus. Very few texts, however, use determinants at all consistently. To do so would necessitate the addition of three or four theorems to the chapter in algebra. And while I have no settled opinion in this matter, I believe it would be advantageous for the Association to discuss at some future meeting whether or not a more consistent use of this invaluable mathematical tool in elementary work is desirable.

It is essential that one be able to solve a quadratic equation by completing the square, for this process is frequently used in other connections. As regards simultaneous quadratics, the case in which one equation is linear is of first importance, and the only others which are important are pairs of equations such that the elimination of one variable may be effected by obvious methods. The usual method for treating symmetrical

equations is occasionally convenient but not essential, and the so-called homogeneous quadratics, which are usually not homogeneous, are of no use.

Too great emphasis cannot be laid upon the fundamental importance of the determination of the nature of the roots of a quadratic by means of the discriminant, and the expressions for the sum and product of the roots in terms of the coefficients may be used occasionally.

Of course the work in analytics proceeds much more readily if the student has made free use of graphs. So much has been said concerning graphs in recent years that I shall pass over them with the remark that in my opinion not even our most recent texts have made as much use of graphs as is desirable for a complete understanding of elementary subjects, quite apart from the advantages analytics and calculus would derive from a more extended use of graphic methods. But the use of graphs in algebra should be carefully restricted to the clarifying of the concepts of algebra, unless, indeed, and here we come in contact with an entirely different question, it is desirable to correlate certain portions of analytics with the more elementary work. I mention this explicitly because I have heard the derivation of the equations of conic sections from geometric definitions discussed as a part of graphic algebra.

Before leaving the subject of algebra permit me to emphasize the important topics negatively. The student of analytics and calculus has occasional use for the binomial theorem, for the rule for squaring the sum of several terms, for a continued proportion, for the sign of a quadratic (best determined, as is the sign of any function, from its graph), for the limit of a fraction as a variable becomes infinite, for the meaning of variation, and for the facts that a polynomial of degree n may be written as the product of n linear factors, that imaginary roots of a polynomial with real coefficients occur in conjugate pairs, and that two roots of a polynomial become infinite when and only when the coefficients of the two highest powers of the variable approach zero. But I do not believe that his work would be seriously hampered by not having had these things in advance.

I have found use for the rule for the extraction of the real cube root of a number but once since I studied it, and on that

occasion a student then studying algebra asked me to explain it. Nor do I find any use for the extraction of square and cube roots of polynomials, except such as are obviously perfect squares or cubes. Equations in which the reciprocals of x, y, z are treated as unknowns do not arise, and one sees only very simple equations in the quadratic form. But if equations in the quadratic form are solved by setting a single variable for a function, the student becomes acquainted with one of the most fruitful methods of mathematics, namely, change of variable. The idea of ratio is fundamental but the technique of proportion is not. There is little use for arithmetic and none for harmonic progressions. The infinite geometric series is useful, but with the possible exception of this series, I believe that the whole topic of infinite series is best left to the calculus. The binomial theorem for fractional exponent, the exponential and logarithmic series are derived so simply by Taylor's formula that it seems to me a useless waste of time to derive them by algebraic methods. And unless the latter is done the mere question of convergence of an infinite series is hardly worth considering.

The texts in analytics and calculus make little use of the theory of equations except as already indicated, but they might very well do so. Permutations and combinations are of no importance, and as the first course in analytics and calculus deals only with real numbers, no intimate knowledge of complex numbers is necessary. Partial fractions are used in a fundamental problem of the integral calculus, but by the time a student needs them his algebraic technique is sufficiently developed to handle them readily. I say this with assurance because I have taught the calculus to a few classes whose preparation had not included partial fractions.

Plane trigonometry may be dismissed with the remark that all of the usual course is essential, except the formulas for the solution of oblique triangles for which there is little or no use. The importance of the solution of the right triangle and of all of the analytic side of the subject is inestimable.

Comparatively few theorems of plane geometry are actually used, but a general knowledge of the subject is essential. In terms of the syllabus of the National Committee of Fifteen, an

adequate preparation would be given by considering the theorems in black face type, in italics and Roman. Some of the latter could undoubtedly be omitted, while of those in small type I have noticed but one for which I recall a use, namely, that a circle is determined by three points. But the omission of this theorem would cause no difficulty. Even fewer theorems of solid geometry are needed. Except for the rules for the areas and volumes of prisms, pyramids, cylinders, cones and spheres (but not for truncated solids, frustums, spherical pyramids, etc.) the student needs only an idea of how things look in space, or in other words, only facts which are intuitively evident and which may be stated in terms easily understood.

One topic, algebraic in its nature but applied chiefly in geometry, namely, the theory of limits, is used fundamentally in the calculus. But in recent years there has been a decided leaning toward the belief that a formal treatment of this subject is unsuitable for high school students and that it is best postponed until later.

Thus many portions of high school mathematics are not used explicitly in the study of analytics and calculus. These subjects are taught in Yale College without being preceded by a course in advanced algebra and in Princeton and Dartmouth to students who have not had solid geometry. I do not approve of the plan at Yale. And if the Princeton-Dartmouth scheme is worth consideration the discussion must not be of the question, Can solid geometry be discarded? but rather, Can we replace solid geometry by something more valuable?

What is needed for the problems of analytics and the calculus is a considerable knowledge of geometry for their proper orientation, and for their solution, the solution of equations and the transformation of functions, both algebraic and transcendental.

In conclusion, let me say most emphatically that the development of mathematical maturity and technique sufficient for the successful pursuit of analytic geometry and the calculus requires more than a study of the facts there used. Whatever may be the topics the student pursues, he needs at least as much mathematical training as he now receives.

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